C. Handlebodies

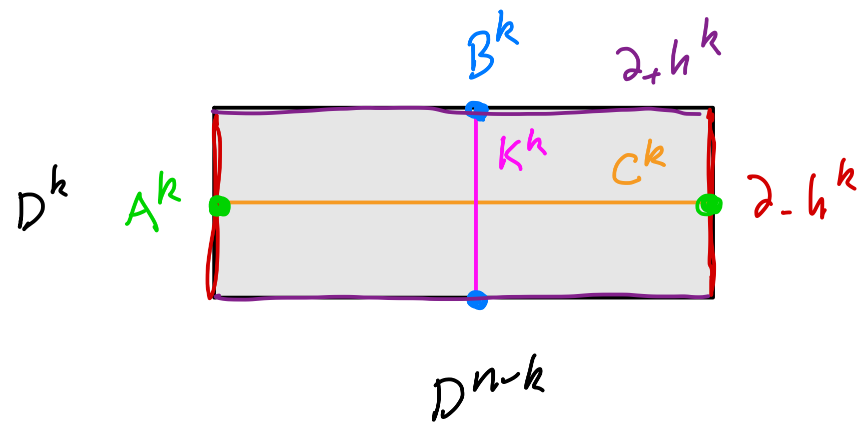
index of handle

an n-dimensional k-handle is

$$h^k = D^k \times D^{n-k}$$

set

	$\partial_- h^k = (\partial D^k) \times D^{n-k}$	attaching region
	$\partial_+ h^k = D^k \times (\partial D^{n-k})$	
	$A^k = (\partial D^k) \times \{0\}$	attaching sphere
	$C^k = D^k \times \{0\}$	core
	$B^k = \{0\} \times (\partial D^{n-k})$	belt sphere
	$K^k = \{0\} \times D^{n-k}$	co-core

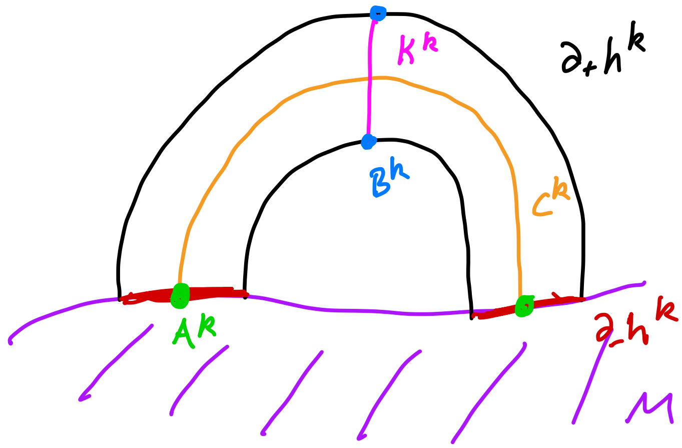


given an n-manifold M and an embedding

$$\phi : \partial_- h^k \to \partial M$$

we attach h^k to M by forming the quotient space

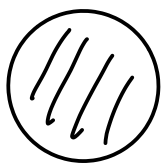
$$M \amalg h^k \Big/ (x \in \partial_- h^k) \sim (\phi(x) \in \partial M)$$



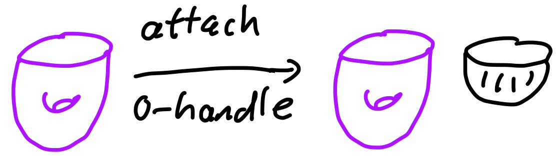
example:

in dimension 2:

<u>k = 0:</u>



so



$\partial_- h^0 = \emptyset$

<u>k = 1:</u>

h^1

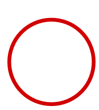
$\partial_- h^1 =$

so

<u>k = 2:</u>

h^2

$\partial_- h^2 =$



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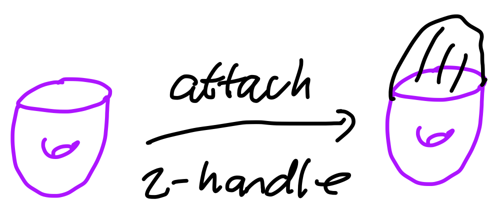


Remark:

1) In all dimensions attaching a 0-handle is just disjoint union with D^n

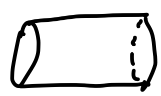
2) <u>only</u> a 0-handle can be attached to

the empty set

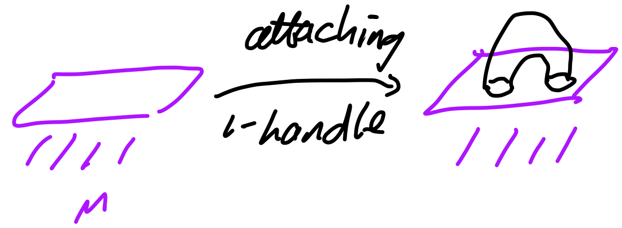
3) In all dimensions n attaching an n-handle is just "capping off" an S^{n-1} boundary component

in dimension 3:

$k=0$: ✓

$k=1$: 

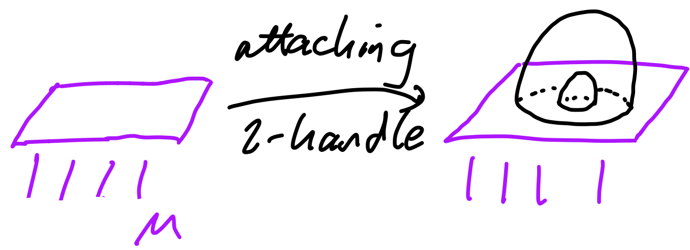
$D^1\times D^2$

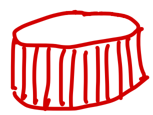
so 

$\partial_- h^1 =$ 

$k=2$: 

$D^2\times D^1$

so 

$\partial_- h^2 =$ 

$k=3$: ✓

Remark: Note that when a handle is attached one has a manifold with "corners". There is a standard way to smooth them to get a manifold with boundary (see Wall "Differential Topology")

lemma 1:

> If $\varphi_0, \varphi_1 : \partial M \to \partial N$ are two diffeomorphisms that are isotopic then $M \cup N /_{(x\in\partial M)\sim(\varphi_0(x)\in\partial N)}$ and $M\cup N/_{(x\in\partial M)\sim(\varphi_1(x)\in\partial N)}$ are diffeomorphic

Proof:

let $\Psi : (\partial M \times [0,1]) \to \partial N$ be the isotopy

so $\Psi(x, i) = \varphi_i(x)$ $\quad i = 0, 1$

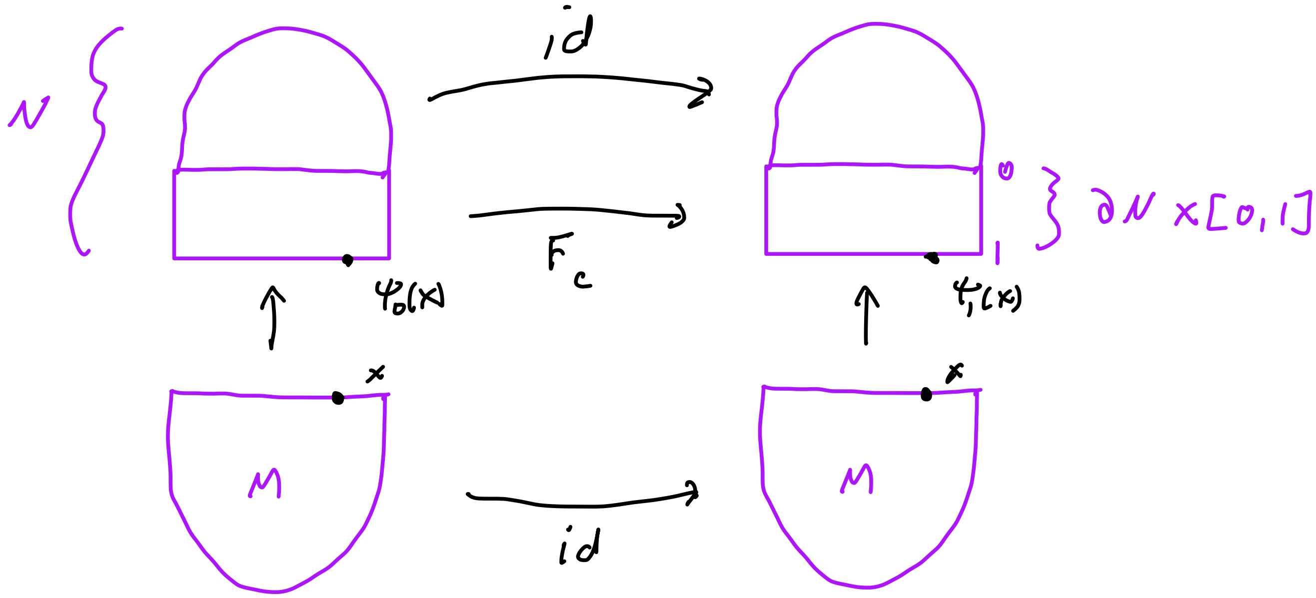
note: $F_c : (\partial N \times [0,1]) \to (\partial N \times [0,1]) : (x,t) \mapsto (\Psi(\varphi_0^{-1}(x), t), t)$

is a diffeomorphism

$F_c(x,0) = x$

$F_c(x,1) = \varphi_1 \circ \varphi_0^{-1}(x)$

now define $\tilde{F}$



note $\tilde{F}$ gives a map F on the quotient spaces

exercise: 1) F is a homeomorphism

2) F is a diffeomorphism

(need to understand smooth structure on quotient space)

We will often use this lemma, but now we need a "relative" version

exercise:

1) if $\phi_0, \phi_1 : \partial_- h^k \to \partial M$ are isotopic, then the result of

attaching h^k to M with ϕ_0 is diffeomorphic to the result of attaching with ϕ_1

2) the isotopy class of $\phi: \partial_- h^k \to \partial M$ is determined by

a) isotopy class of $\phi|_{A^k}$ $\quad (A^k = S^{k-1} \times \{0\})$

(i.e. an S^{k-1} knot in ∂M)

b) the "framing" of the normal bundle of $\phi(A^k)$

i.e. an identification of $\nu(\phi(A^k))$ with $S^{k-1} \times \mathbb{R}^{n-k}$

normal bundle

$T\partial M|_{\phi(A^k)} / T\phi(A^k)$

Hint: Recall $\phi(A^k)$ has a nbhd diffeomorphic to a nbhd of the zero section in $\nu(\phi(A^k))$

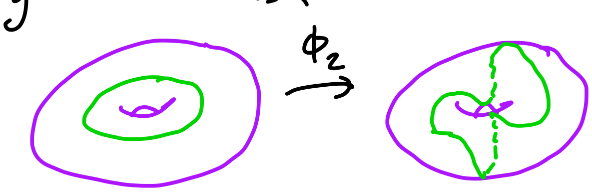
the easiest way to get this is by choosing a Riemannian metric and use the exponential map

now choose metrics cleverly

example: notice $S^1 \times \mathbb{R}^2$ has an integers worth of framings

$$S^1 \times \mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}^2$$
$$(\phi, (r, \theta)) \longmapsto (\phi, (r, \theta + n\phi))$$

eg on unit disk



3) show the framings on a k-dimensional sphere in Y^n are in one-to-one correspondence with $\pi_k(O(n-k))$ dim. of the fiber of normal bundle

from above we see that

$$\boxed{\begin{array}{l}\text{attaching an } n\text{-dimensional } k\text{-handle one must specify} \\ \quad 1) \text{ an } S^{k-1} \text{ knot in } \partial M \text{ and} \\ \quad 2) \text{ an ``element'' of } \pi_{k-1}(O(n-k))\end{array}}$$

to really get an element need a canonical "zero" framing

examples:

1) a 0-handle in an n-manifold

attaching sphere is $\emptyset$ and

framing $\pi_{-1}(O(0)) = \emptyset$

so no choices! unique way to attach a 0-handle

we saw above attaching 0-handle is just taking union with B^n

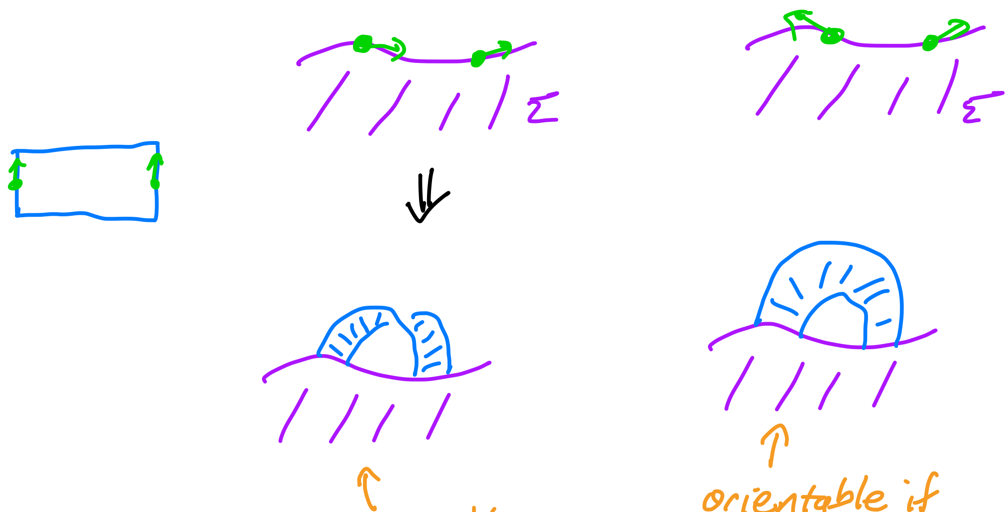
2) 1-handles in an n-manifold

attaching sphere is $S^0 = \cdot \;\; \cdot$

framing $\pi_0(O(n-1)) = \mathbb{Z}/2 \quad (n>1)$

so once you pick 2 points to attach 1-handle there are 2 ways to attach handle

$n=2$



not orientable

orientable if Σ way

this happens in general: when attaching a 1-handle once attaching sphere is specified there is a unique way to attach handle and preserve orientability

(assuming M connected)

3) 2-handles in an n-manifold

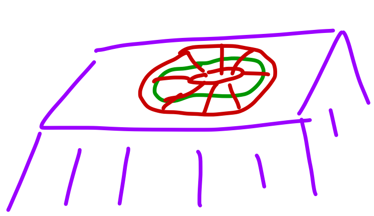
attaching sphere an S^1

framing element of $\pi_1(O(n-2)) \cong \begin{cases} \{e\} & n-2=0,1 \\ \mathbb{Z} & n-2=2 \\ \mathbb{Z}/2 & n-2>2 \end{cases}$

$n=2$

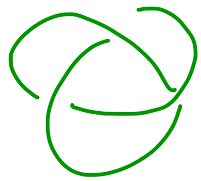
no choice once S^1 specified

$n=3$

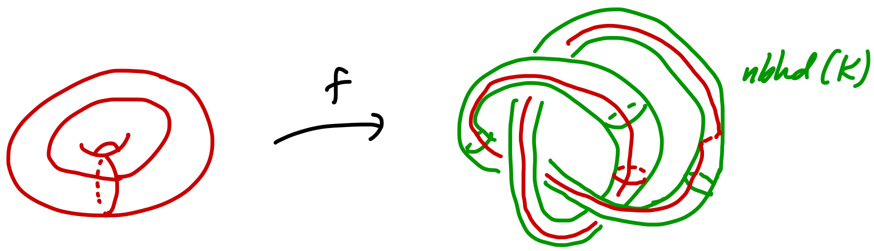


“ ”

$n=4$



attaching sphere
a knot in 3-mfd $= \partial$(4-mfd)



get other "framings" by composing $f \circ \phi_n$

$\phi_n : S^1 \times D^2 \to S^1 \times D^2 : (\phi, r, \theta) \mapsto (\theta, r, \theta + n\phi)$

A handle decomposition of an n-manifold M is a sequence of manifolds $M_0, M_1, \ldots M_\ell$ such that

1) $M_0 = \emptyset$ and $M_\ell \cong M$

2) M_{i+1} is obtained from M_i by a k-handle attachment for some k

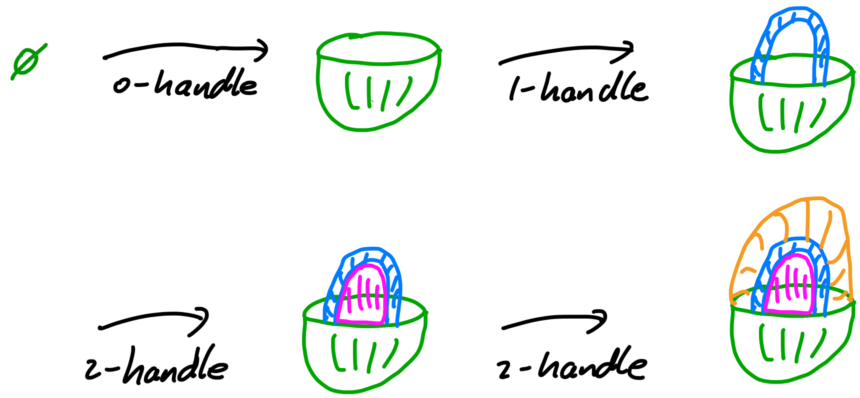
a handle decomposition of a cobordism M with $\partial_- M \neq \emptyset$ is the same except $M_0 \cong [0, \varepsilon] \times \partial_- M$

example:

handle decompositions of S^2

1)

2)



Th$^{\underline{m}}$ 2:

$\boxed{\text{Any smooth compact manifold has a handle decomposition}}$

This follows from the existence of Morse functions!

Main Th$^{\underline{m}}$ of Morse Theory:

let $f: M \to \mathbb{R}$ be a Morse function

I) if $[a,b]$ contains no critical values then

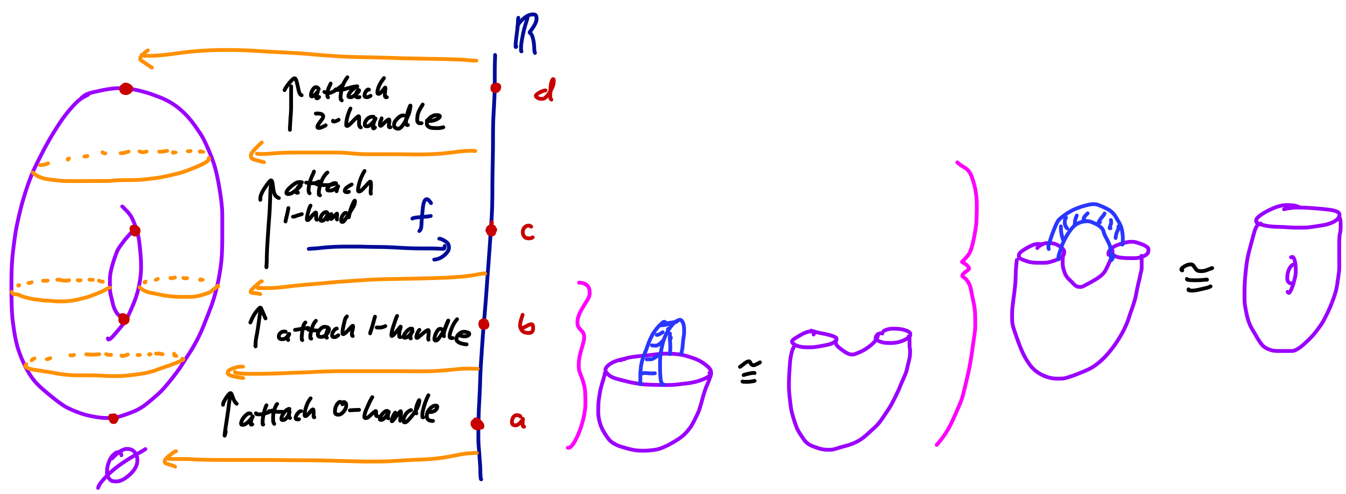
$$f^{-1}([a,b]) = f^{-1}(a) \times [a,b]$$

manifold since a regular value

II) if $\exists!$ critical point $p \in f^{-1}([a,b])$ of index k s.t. $f(p) \in (a,b)$ then

$f^{-1}([a,b])$ is obtained from $f^{-1}(a) \times [a, a+\varepsilon]$ by attaching a k-handle to $f^{-1}(a) \times \{a+\varepsilon\}$

example:



Remark: handle decomposition theorem clearly follows

Proof of I:

recall a Riemannian metric on M is a choice of "inner product" on T_xM for each x

i.e. $g_x : T_xM \times T_xM \to \mathbb{R}$

such that
1) g_x is bilinear
2) $g_x(v,w) = g_x(w,v)$ and
3) $g_x(v,v) \geq 0$ and
$g_x(v,v) = 0 \Leftrightarrow v = 0$

and g_x variès smoothly with x
(technically g is a positive-definite, symmetric 2-tensor)

Fact: all manifolds have Riemanian metrics

given g on M we get a map

$$\phi_g : \mathfrak{X}(M) \longrightarrow \Omega^1(M) : v \longmapsto g(v, \cdot)$$

vector fields

1-forms

<u>exercise:</u> show ϕ_g is an isomorphism

given a function $f : M \to \mathbb{R}$

we define the gradient to be

$$\nabla f = \phi_g^{-1}(df)$$

<u>exercise:</u> Show ∇f_x is perpendicular to $f^{-1}(f(x))$ if $f(x)$ a regular value

note $\neq 0$ since $df \neq 0$

now set

$$v = \begin{cases} \nabla f / \|\nabla f\|^2 & \text{near } f^{-1}([a,b]) \\ 0 & \text{outside a nbhd of } f^{-1}([a,b]) \end{cases}$$

$g(\nabla f, \nabla f)$

let $\Phi : M \times \mathbb{R} \to M$ be the flow of v

recall $\phi_t : M \to M : x \longmapsto \Phi(x,t)$ is a diffeomorphism

<u>note:</u>

$$\begin{aligned} \frac{d}{dt}\left(f \circ \Phi(x,t)\right) &= df_{\Phi(x,t)}\left(\frac{d}{dt}\Phi(x,t)\right) \\ &= df_{\Phi(x,t)}(v) \\ &= g\left(\nabla f, \frac{\nabla f}{\|\nabla f\|^2}\right) = 1 \end{aligned}$$

so $f \circ \Phi(x,t) = t + \text{constant} = t + f(x)$

$f(x)$

(in a nbhd of $f^{-1}([a,b])$)

$\therefore \quad \phi_t(f^{-1}(a)) = f^{-1}(a+t) \qquad t \in [0, b-a]$

so $\phi_t|_{f^{-1}(a)} : f^{-1}(a) \to f^{-1}(a+t)$

is a diffeomorphism

now set

$$\Psi : f^{-1}(a) \times [0, b-a] \to M$$
$$(x,t) \longmapsto \phi_t(x) = \Phi(x,t)$$

<u>Claim</u>: $d\Psi_{(x,t)}$ an isomorphism

indeed

if $(v, c\frac{\partial}{\partial t}) \quad T_{(x,t)}(f^{-1}(a) \times [0, b-a])$

then $d\Psi_{(x,t)}(v, c\frac{\partial}{\partial t}) = d\Phi_{(x,t)}(v, c\frac{\partial}{\partial t})$

$$= \underbrace{d\phi_t(v)}_{\text{in } T_{\phi_t(p)}(f^{-1}(a))} + c\underbrace{\frac{\nabla f}{\|\nabla f\|^2}}_{\text{perpendicular to}}$$

so if $d\Psi_{(x,t)}(v, c\frac{\partial}{\partial t}) = 0$ then each component is 0

$\Rightarrow v = 0$ since ϕ_t a diffeo

$c = 0$ since $\nabla f \neq 0$ ✓

Clearly: Ψ is onto (exercise if not obvious)

Claim: Ψ injective

if $\Psi(x,t) = \Psi(y,s)$ then $t = s$

so $\phi_t(x) = \phi_s(y) = \phi_t(y)$

$\therefore x = y$ ✓

a bijective local diffeomorphism is a diffeomorphism ▱

Sketch of Part II:

First we need

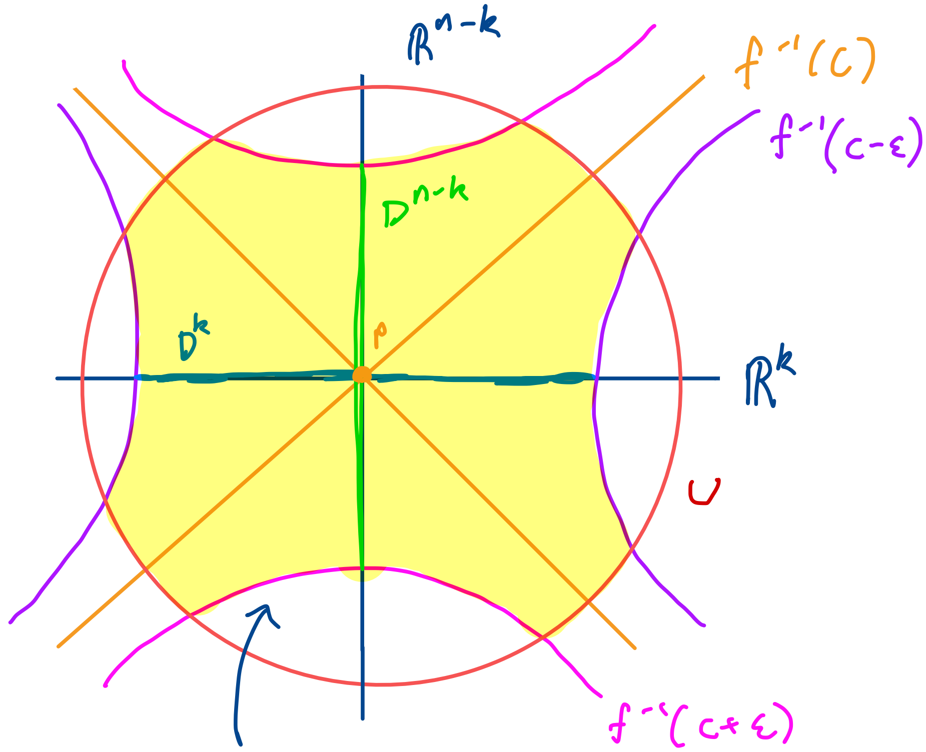
lemma 3 (Fundamental lemma of Morse theory):

> If p is a non-degenerate critical point of $f: M \to \mathbb{R}$ with index k then $\exists$ coordinates about p such that f takes the form
>
> $$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

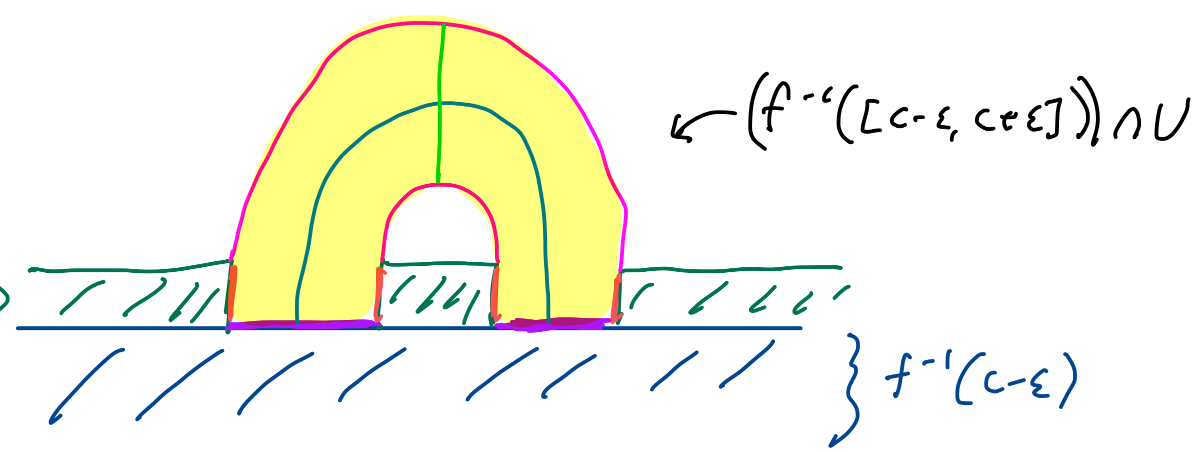
will prove this later but first see how Part II)
follows from the lemma

let U be a neighborhood of the critical point p
as in the lemma

so in U we see (let $f(p) = c$)



this is essentially k-handle attachment



$$\overline{f^{-1}([c-\varepsilon, c+\varepsilon]) - U} \cong \left(\overline{f^{-1}(c-\varepsilon) - U}\right) \times I$$

note this doesn't change topology

Proof of lemma 3:

take any coordinate chart φ sending 0 to p, writing f in these coordinants (i.e. $f \circ \varphi$ but we just write f)

$$\left(\frac{\partial^2}{\partial x_i \partial x_j} f\Big|_0\right)$$

can be diagonalized by an appropriate choice of basis.

If $L : \mathbb{R}^n \to \mathbb{R}^n$ is this change of basis map then replace ψ by $\psi \circ L$ (still call it ψ), then f in these coords

$$A = \left(\frac{\partial^2}{\partial x_i \partial x_j} f\Big|_0\right) = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & +1 & & \\ & & & & \ddots & \\ & & & & & +1 \end{pmatrix}$$

to further normalize $f \circ \psi$ we write in is a special way

from here on we write f for $f \circ \psi$ and just think of f as a function on an open set in $\mathbb{R}^n$

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx)\, dt \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, x_i \, dt \\ &= \sum_{i,j=1}^n \underbrace{\int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(st\, x)\, dt\, ds}_{b_{ij}(x)}\; x_i x_j \\ &= \sum_{i,j=1}^n b_{ij}(x)\, x_i x_j = x^T B_x\, x \end{aligned}$$

where $B_x = (b_{ij}(x))$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

by replacing b_{ij} with $\frac{1}{2}(b_{ij} + b_{ji})$ we can assume B_x is symmetric

could redo above to see $f = x^T B_x^T x$ too

<u>note</u>: $B_0 = A$

<u>Claim</u>: we can find an invertable matrix Q_x depending on x such that $Q_x^T B_x Q_x = A$

(e.g. $Q_0 = I$)

then set $\psi(x) = Q_x^{-1} x$ and note

$$d\psi_0 = Q_0^{-1} = I$$

so ψ is a local diffeo (coord. chart!)

$$\psi : \underset{(x_1, \dots, x_n)}{V} \to \underset{(y_1, \dots, y_n)}{U}$$

and $f(y) = f(0) + (Q_{\psi^{-1}(y)}y)^T B_{\psi^{-1}(y)} (Q_{\psi^{-1}(y)} y)$

$$= f(0) + y^T \left(Q_{\psi^{-1}(y)} B_{\psi^{-1}(y)} Q_{\psi^{-1}(y)}\right) y = f(0) + y^T A y$$

$\therefore$ if we set $\phi = \psi^{-1}$, then

$$\begin{aligned} f\circ\phi(y) &= f(0) + \psi(\phi(y))^t A\, \psi(\phi(y)) \\ &= f(x) + y^t A y \\ &= f(x) - y_1^2 - \dots - y_h^2 + y_{h+1}^2 + \dots y_n^2 \quad \blacksquare \end{aligned}$$

Proof of Claim: We show, given $A = \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix}$

$\exists$ a nbhd N of A in space of $n \times n$ matricies

and a smooth map $P : N \to GL(n, \mathbb{R})$

s.t. $P(A) = I$ and $P(B)^t B P(B) = A \quad \forall B \in N$

to see this suppose B is close enough to A so that

$b_{11} \neq 0$ and has the same sign as a_{11}

consider the map $T : \mathbb{R}^n \to \mathbb{R}^n$ given by
$y \longmapsto x$

$$T = \sqrt{\frac{|a_{11}|}{|b_{11}|}} \begin{pmatrix} 1 & -b_{12}/b_{11} & \cdots & -b_{1n}/b_{11} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

note:

$$T^t B T = \sqrt{\frac{|a_{11}|}{|b_{11}|}}\, T^t \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{12} & & & \\ \vdots & & B' & \\ b_{1n} & & & \end{pmatrix}$$

$$= \frac{|a_{11}|}{|b_{11}|}\begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B'' & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B''' & \\ 0 & & & \end{pmatrix}$$

and if $b_{11} \sim \pm 1$, $b_{1i} \sim 0$ then B''' close to A_{11} so we can induct

minor

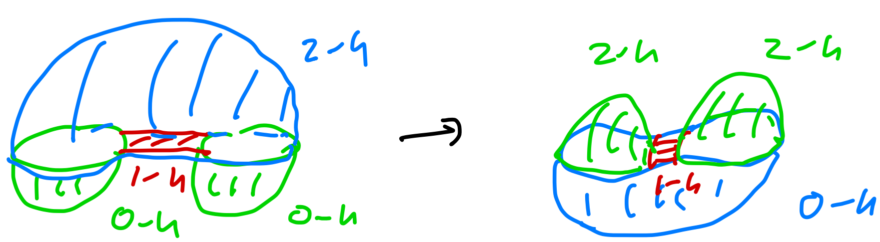
D. Main Facts about handle decompositions

lemma 4:

> in dimension n, an "upside down" k-handle is an $(n-k)$ handle
>
> ie. if you attach a k-handle to $M \times \{1\} \subset M \times [0,1]$ to get W, where $\partial W = -M \cup M'$
> then W is diffeomorphic to the result of attaching an $(n-k)$-handle to $M \times \{0\} \subset M \times [0,1]$

Remark: if a handle decomposition of M came from a Morse function $f: M \to \mathbb{R}$
then $-f: M \to \mathbb{R}$ a Morse function s.t.
an index k critical point for f becomes and index $n-k$ critical point of $-f$

example

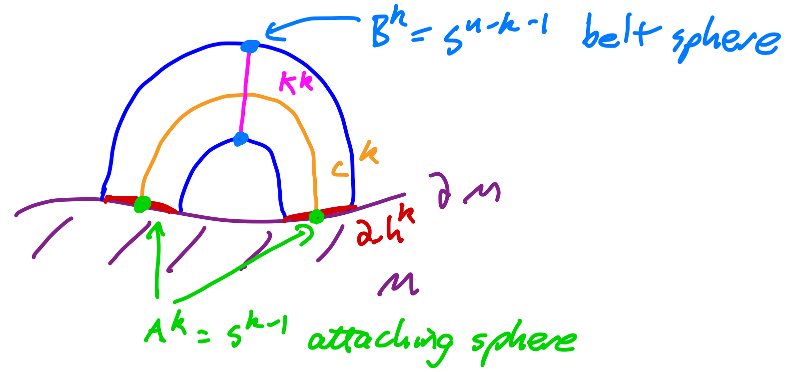


exercise: Prove lemma

lemma 5:

> Given a handlebody structure on M we may always assume handles of index k are attached <u>before</u> handles of index $\ell > k$ and k-handles can be attached in any order

Proof: recall



now suppose h^k is attached after h^ℓ $\ell \geq k$

note:

$$\begin{array}{l} \text{for } h^k \to A^k = S^{k-1} \subset \\ \text{for } h^\ell \to B^\ell = S^{n-\ell-1} \subset \end{array} \underbrace{\partial(M \cup h^\ell)}_{n-1 \text{ manifold}}$$

we can isotop A to be transverse to B

recall isotopy does not affect the resulting mfd

so $A \cap B$ has dimension

(codimension adds)

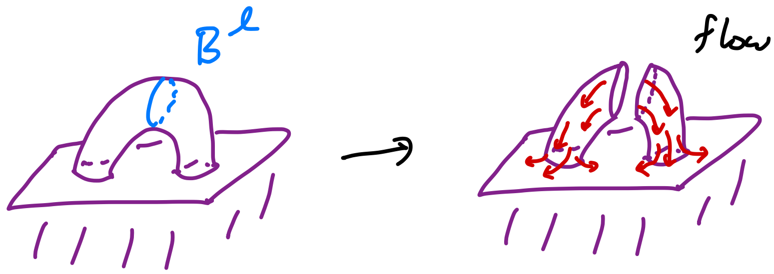
$$(n-1) - (n-k+1) - (n-n+\ell-1)$$

$$= k - \ell - 1 < 0$$

so A is disjoint from B

exercise: if $f: N \to M \cup h^\ell$ is an embedding disjoint from B^ℓ, then f can be isotoped so $\mathrm{im}(f)$ is disjoint from h^ℓ

hint:



flow that pushes anything in $\partial_+ h^\ell - B$ off of $\partial_+ h^\ell$

$\therefore$ the attaching region of h^k can be moved away from h^ℓ and so h^k and h^ℓ can be attached in any order

note: Not true for $k > \ell$

h^1 cannot come before h^2

so all k handles can be attached a the same time but there is an interesting thing that happes when you do this

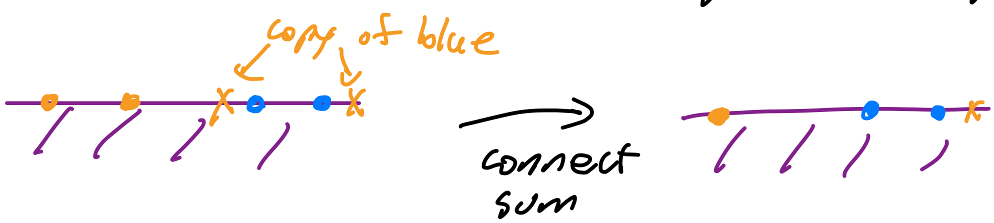
suppose you had

now you push orange of of the blue in 2 ways

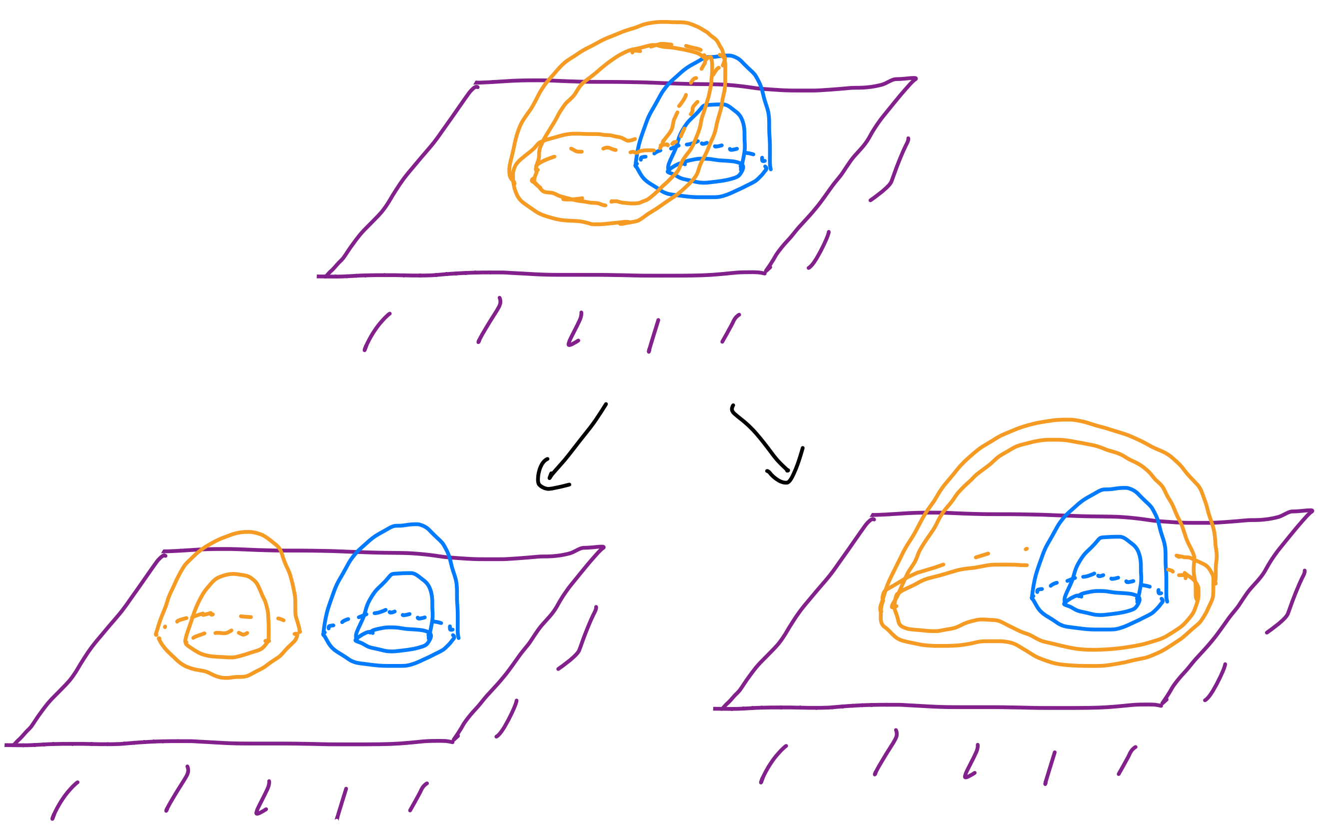
the difference is called a <u>handle slide</u>

<u>note:</u> the attaching spheres change as follows

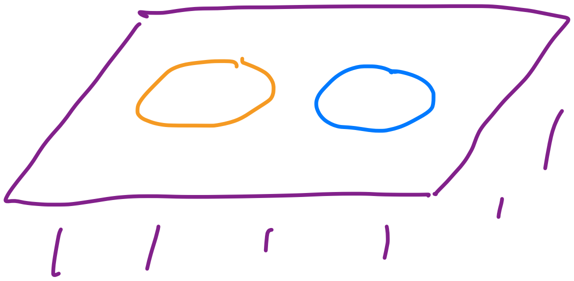
you "make a copy of blue then
connect sum the orange with copy"



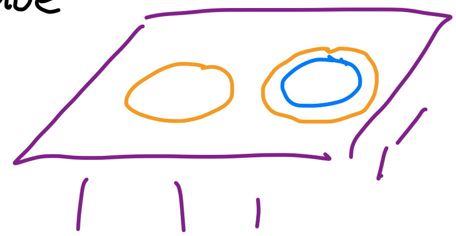
this is easier to see with 2-handles in 3D



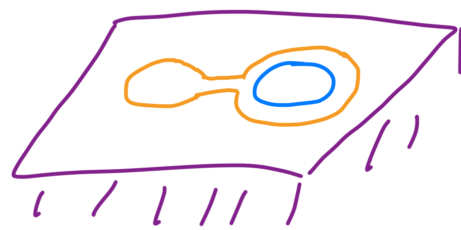
so attaching spheres are



make a copy of blue



connect sum



$D^h \times D^{n-h}$

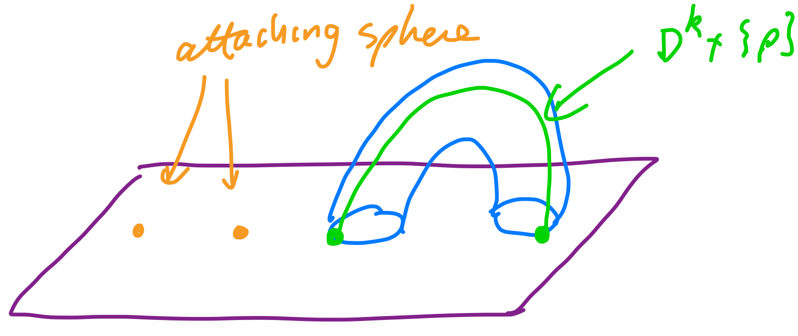
the way to prove this works in any dimension is as follows

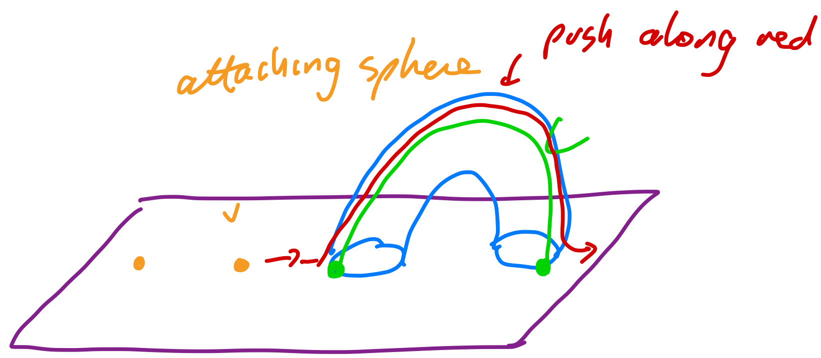
1) in blue handle you see $D^k \times \{p\} \subset \partial_+ h^k$

it gives a $S^{k-1} \subset \partial M$ (copy of attaching sphere)

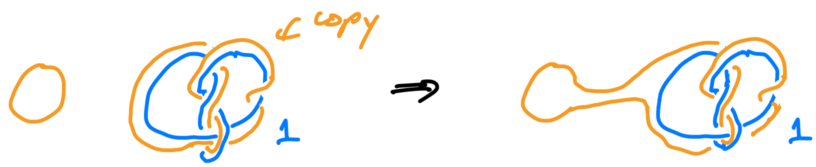
2) can move part of orange attaching sphere near this S^{k-1} and use D^k to guide an isotopy over blue handle

e.g.





note: even through we can't "see" a 4-dimensional 2-handle we can see how there attaching spheres change as we do a handle slide

eg: 

we will discuss this much more later on in the course.

lemma 6:

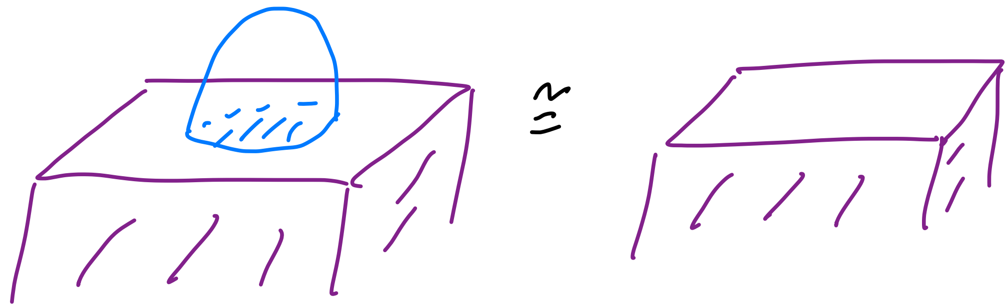
If $M' = M \cup h^k \cup h^{k+1}$ such that the attaching sphere of h^{k+1} intersets the belt sphere of h^k exactly once and transversely, then $M' \cong M$

we say the handles can be cancelled

Proof: we need a preliminary result

lemma 7:

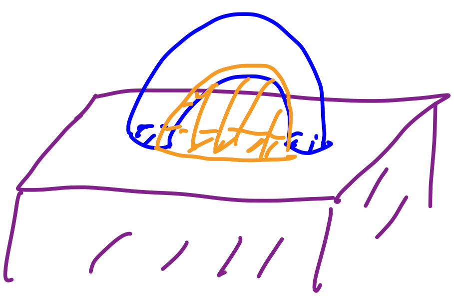
if M is an n-manifold with boundary A = disk $D^{n-1} \subset \partial D^n$, and $f: A \to \partial M$ an embedding then $M \cup_f D^n \cong M$



exercise:

1) Prove lemma 7

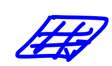
2) Under the hypothesis of lemma 6 show you can isotop attaching region of h^{k+1} so $h^k \cap h^{k+1}$ is a disk $\therefore$ lemma 7 $\Rightarrow h^k \cap h^{k+1} \cong D^n$



3) Show $((h^k \cup h^{k+1}) = D^n) \cap M$ is a disk $\therefore$ lemma 7 $\Rightarrow M' \cong M$

so $h^k \cap h^{k+1} =$ 

$M \cap (h^k \cup h^{k+1}) =$ 



Corollary 8:

if W is a connected n-dimensional cobordism then it has a handle decomposition with

$$\left.\begin{matrix}1\\0\end{matrix}\right\} \text{ 0-handles if } \partial_- W \begin{cases} = \emptyset \\ \neq \emptyset \end{cases}$$

$$\left.\begin{matrix}1\\0\end{matrix}\right\} \text{ n-handles if } \partial_+ W \begin{cases} = \emptyset \\ \neq \emptyset \end{cases}$$

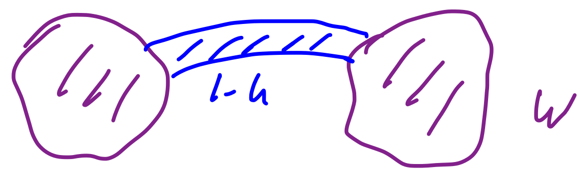
Proof:

note: • if $k > 1$, then $\partial_- h^k$ is connected so attaching h^k must be done to a component of W

$\therefore W \cup h^k$ has the same number of components as M

• attaching a 0-handle adds components to W

• attaching a 1-handle either keeps the number of components the same or reduces the number by one



now assume $\partial_- W = \emptyset$, we can attach all zero handles first

if there is more than one, then W will not be connected unless $\exists$ 1-handle h^1 connecting 2 of the 0-handles h_1^0, h_2^0

note: belt sphere $= \partial h_i^0$

attaching sphere of $h^1 = S^0$

so by Fact 3 above can cancel $h^1 \cup h_2^0$

note: given a Morse function f, $-f$ is Morse too with same critical points but if one was index k for f then its $n-k$ for $-f$

similarly if you think of a handlebody "upside down" you have same handles but k handle becomes $n-k$

$\therefore$ also done with $\partial_+ W = \emptyset$

exercise: do other cases